turbation technique, based on assuming a solution in the form

$$x = \sum_{i=1}^{\infty} e_0^i x_i \qquad z = \sum_{i=1}^{\infty} e_0^i z_i$$

To the first order in e_0, the result is

$$x = 2C + A\cos v_0 + B\sin v_0 + e_0(-\tfrac{3}{2}A + A\cos v_0 - B\sin v_0 + \tfrac{1}{2}A\cos 2v_0 + \tfrac{1}{2}B\sin 2v_0 - 3Cv_0\sin v_0)$$

$$y = D - 3Cv_0 - 2A\sin v_0 + 2B\cos v_0 + e_0(3Av_0 - 2A\sin v_0 + 2B\cos v_0 - \tfrac{1}{2}A\sin 2v_0 + \tfrac{1}{2}B\cos 2v_0 + 6C\sin v_0 - 6v_0\sin v_0)$$

$$z = E\cos v_0 + F\sin v_0 + e_0[\tfrac{1}{2}E - \tfrac{1}{6}E\cos v_0 - \tfrac{1}{3}E\cos 2v_0 + \tfrac{1}{3}F\sin v_0 - \tfrac{1}{6}F\sin 2v_0]$$

Note the appearance of "secular" terms with v_0, $v_0 \cos v_0$, and $v_0 \sin v_0$ and the appearance of higher harmonics in the terms that have e_0 as factor. The secular terms are of course unavoidable since they indicate the continuously growing relative distance between two points in close orbits of slightly different period. It is interesting to note that no terms with v_0^2 appear, even if the solution is carried out to include e_0^2. It is not surprising that the integration constant C is closely connected with the semimajor axis; it is, on the other hand, somewhat surprising that very simple relations exist between the other integration constants and elliptic parameters. In terms of the initial condition of relative position and velocity, the integration constants are

$$A = -(3x(0) + 2y'(0)) \qquad D = y(0) - 2x'(0)$$
$$B = x'(0) \qquad E = z(0)$$
$$C = 2x(0) + y'(0) \qquad F = z'(0)$$

A comparison with Eqs. (14) shows that the integration constants are precisely the changes in the orbital elements at least as far as first-order terms in the relative positions and velocities are concerned.

Reference

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A Note on Lunar Librations

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ALTHOUGH it is considered that gravity-gradient torque is insufficient by itself for attitude control of artificial satellites, the mechanically equivalent phenomenon of lunar librations is nevertheless of great technical interest. Recent studies of the satellite problem disclose basic features of the motion, equally valid for the moon, but not elucidated in the specialized literature on that subject. Aside from the fundamental interest in a classic problem and possible new basis for interpretation of amassed observational data, the results are important for the newer stability analysis.

The fact that the moon persistently presents the same face toward earth, enunciated more than two centuries ago as Cassini's first law, stimulated researches by Lagrange, Laplace, and many others. Departures from this idealized motion, termed physical librations, are of such small amplitude as to have thwarted all attempts to measure them astronomically, a fact the more remarkable in view of the nearly perfect symmetry of lunar mass distribution. From the purely physical viewpoint, nothing less than mathematical proof is needed to lend plausibility to an obvious fact so much at variance with intuition.

Librational motion about a mass center itself in nonuniform motion is described by Euler's equations of rigid-body motion, extended to include effects of relative motion. This system of three coupled partial differential equations for the lunar motion, strictly nonlinear, has classically been treated by studying motion slightly perturbed from equilibrium and by further limiting the analysis to the longitudinal motion that is then uncoupled.[3] The remaining two modes, termed physical libration in inclination and physical libration in node, have meanwhile been essentially neglected. Although these two strongly coupled modes appear at first sight to be the most formidable ones from the mathematical standpoint, it will now be shown that important properties of these modes are revealed by applying directly the more detailed treatments given in satellite studies. In addition, these characteristics strongly suggest that the traditional preference for isolating longitudinal motion was an unfortunate choice made long ago and not corrected by later workers.

Free lunar librations for idealized Keplerian motion in a circular orbit are governed by equations given in Ref. 1; with unimportant changes of notation to conform with standard usage in the literature on that subject, these are

$$\ddot{\alpha} + \Omega^2\left(\frac{C-B}{A}\right)\alpha - \Omega\left(1 - \frac{C-B}{A}\right)\dot{\beta} = 0 \qquad (1)$$

$$\ddot{\beta} + 4\Omega^2\left(\frac{C-A}{B}\right)\beta + \Omega\left(1 - \frac{C-A}{B}\right)\dot{\alpha} = 0 \qquad (2)$$

$$\ddot{\gamma} + 3\Omega^2\left(\frac{B-A}{C}\right)\gamma = 0 \qquad (3)$$

Here A,B,C and α,β,γ denote, respectively, principal inertia moments and small angular displacements from equilibrium for nodal (i.e., earth-pointing), inclination (i.e., moon latitude), and longitudinal components of the physical libration, and Ω is lunar orbital angular speed. The nearly symmetrical mass distribution is shown by the smallness of the dimensionless inertia differences, which have numerical values given by (see, e.g., Ref. 2)

$$(C-A)/C = 0.000627 \qquad (B-A)/C = 0.000118 \qquad (4)$$

Denoting these for convenience by ϵ_3 and ϵ_1, respectively, the third one of the differences that appear in the system of equations, denoted by ϵ_2, is closely obtained as the difference $\epsilon_3 - \epsilon_1$; these three quantities then satisfy the important inequalities

$$0 < \epsilon_1 < \epsilon_2 < \epsilon_3 \qquad (5)$$

Equation (3) shows that longitudinal libration is uncoupled from the other modes, with period of free oscillation inversely proportional to the square root of ϵ_1; its numerical value is about 53 months. This is the part of the motion examined theoretically and sought unsuccessfully through observations started by Bessel more than 100 years ago. Principal attention in modern times has centered around the forced motion resulting from solar attraction and orbital ellipticity.

Physical librations in node and inclination, described by Eqs. (1) and (2), are obviously strongly coupled by the first derivative terms of order unity. Each equation admits harmonic solutions, and it is found that physical libration in inclination "leads" the nodal motion with a phase angle nearly equal to 90°. Of perhaps even greater importance from standpoint of observation is the fact that one of the two periods of free motion is very much smaller than the other, small even when compared with the period of free longitudinal

Received April 19, 1963.

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motion. One of these is closely given by $1 - \frac{3}{2}\epsilon_3$, where the unit of time is the 1-month lunar orbital period; the other is similarly given by

$$\frac{1}{2\epsilon_3}\left(1 - \frac{\epsilon_1}{\epsilon_3}\right)^{-1/2}$$

Using the numerical values given by (4), these periods correspond very nearly to 1 month and to 870 months, respectively. Whereas the very long-period motion presents obvious obstacles from the point of view of measurements for any significant part of even one oscillation, the shortest-period motion, viz., the coupled nodal-inclination mode of period 1 month, is a distinctly more attractive prospect for observation. The shortness of the period also justifies neglecting effects of long-period forcing functions such as the dominant solar attraction.

In summary, it has been shown that the near-symmetry of lunar mass distribution leads to sharply distinguishable dynamic characteristics, and that the mode of shortest period, almost completely overlooked in the past, is a combined motion in node and inclination, interrelated in an elementary manner. The same distinctions should also prove useful as guides for the construction of Liapunov functions required in the application of direct methods for the nonlinear stability problem, where once again the moon should serve as a shining example in a new class of studies in dynamics.

References

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An Explicit Guidance Concept

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Nomenclature

A,**B**	= thrust integrals, defined specifically by Eq. (1)
F	= thrust vector
J_1,J_2,J_3	= time integrals defined specifically by Eq. (3)
m	= mass
r	= central body radius vector to vehicle
S	= slant range vector from target to vehicle
t	= time
V_j	= effective exhaust velocity
X,Y,Z	= inertial, target centered, Cartesian coordinate system
θ	= coangle between thrust vector and Z axis
μ	= central body gravitational constant = gR^2
ξ	= predicted propellant mass fraction to be consumed during burning
τ	= burning time
ψ	= angle between X axis and projection of **F** in X-Y plane
ω	= mean motion = $\{\mu/[\frac{1}{2}(r_0 + r_t)]^3\}^{1/2}$

Superscript

$(\dot{\ })$	= derivative with respect to time

Subscript

c	= command value
n	= refers to value corresponding to nth step
0	= at time zero
t	= at landing site

This development was presented as an Appendix to Ref. 1 at the ARS Lunar Missions Meeting, Cleveland, Ohio, July 17–19, 1962.

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In the text of Ref. 1, a solution was obtained for the differential equation of motion for a particle in a uniform central force field under the influence of a force **F** and of a mass m. This solution is of the form

$$\begin{aligned}\mathbf{r} &= (\mathbf{r}_0 - \mathbf{B})\cos\omega\tau + (1/\omega)(\dot{\mathbf{r}}_0 + \omega\mathbf{A})\sin\omega\tau \\ \dot{\mathbf{r}} &= (\dot{\mathbf{r}}_0 + \omega\mathbf{A})\cos\omega\tau - \omega(\mathbf{r}_0 - \mathbf{B})\sin\omega\tau\end{aligned} \quad (1)$$

where

$$\mathbf{A} = \frac{1}{\omega}\int_0^\tau \frac{\mathbf{F}}{m}\cos\omega t\, dt$$

$$\mathbf{B} = \frac{1}{\omega}\int_0^\tau \frac{\mathbf{F}}{m}\sin\omega t\, dt$$

The components of the thrust vector **F** are assumed to have the following time-dependent form:

$$\begin{aligned}F_x(t) &= F(t)\cos(\theta + \dot{\theta}t)\cos(\psi + \dot{\psi}t) \\ F_y(t) &= F(t)\cos(\theta + \dot{\theta}t)\sin(\psi + \dot{\psi}t) \\ F_z(t) &= F(t)\sin(\theta + \dot{\theta}t)\end{aligned} \quad (2)$$

If the thrust is assumed to be constant and $\dot{\theta}$ and $\dot{\psi}$ are assumed to be of the same order as ω, then analytic solutions for **A** and **B** may be achieved. These are given below as Eqs. (3), correct to first order in $\omega\tau$:

$$\left.\begin{aligned}A_x &= (1/\omega)[J_1\cos\theta\cos\psi - J_2(\dot{\theta}\sin\theta\cos\psi + \dot{\psi}\sin\psi\cos\theta)] \\ A_y &= 1/\omega[J_1\cos\theta\sin\psi - J_2(\dot{\theta}\sin\theta\sin\psi - \dot{\psi}\cos\psi\cos\theta)] \\ A_z &= (1/\omega)(J_1\sin\theta + J_2\dot{\theta}\cos\theta) \\ B_x &= J_2\cos\theta\cos\psi - J_3(\dot{\theta}\sin\theta\cos\psi + \dot{\psi}\sin\psi\cos\theta) \\ B_y &= J_2\cos\theta\sin\psi - J_3(\dot{\theta}\sin\theta\sin\psi - \dot{\psi}\cos\psi\cos\theta) \\ B_z &= J_2\sin\theta + J_3\dot{\theta}\cos\theta \\ J_1 &= -V_j\ln(1 - \xi) \\ J_2 &= (\tau/\xi)(J_1 - V_j\xi) \\ J_3 &= (\tau/\xi)(J_2 - \tfrac{1}{2}V_j\tau\xi)\end{aligned}\right\} \quad (3)$$

where

$$V_j = F/\dot{m} \qquad \xi = \dot{m}\tau/m$$

It is now possible to solve explicitly for $\theta_c, \dot{\theta}_c, \psi_c$, and $\dot{\psi}_c$. The procedure is as follows. Let there be specified some final position and velocity vector, $\mathbf{r}_t$ and $\dot{\mathbf{r}}_t$, which is to be achieved at time = τ. These shall satisfy Eqs. (1). With this substitution, rearrangement yields

$$\begin{aligned}\omega\left(\frac{\mathbf{r}_t}{\cos\omega\tau} - \mathbf{r}\right) &= -\mathbf{B}\omega + (\dot{\mathbf{r}} + \omega\mathbf{A})\tan\omega\tau \\ \left(\frac{\dot{\mathbf{r}}_t}{\cos\omega\tau} - \dot{\mathbf{r}}\right) &= \omega\mathbf{A} + \omega(\mathbf{B} - \mathbf{r})\tan\omega\tau\end{aligned} \quad (4)$$

where, for convenience, the subscript 0 has been dropped.

Equations (4) yield solutions for **A** and **B** as follows:

$$\begin{aligned}\mathbf{A} &= \mathbf{r}_t\sin\omega\tau - 1/\omega\Delta\dot{\mathbf{r}} \\ \mathbf{B} &= \Delta\mathbf{r} + \dot{\mathbf{r}}_t/\omega\sin\omega\tau\end{aligned} \quad (5)$$

where

$$\Delta\mathbf{r} = \mathbf{r} - \mathbf{r}_t\cos\omega\tau \qquad \Delta\dot{\mathbf{r}} = \dot{\mathbf{r}} - \dot{\mathbf{r}}_t\cos\omega\tau$$

The values attained from Eqs. (5) for **A** and **B**, of necessity,